

The supercritical Galton–Watson process in varying environments — Seneta–Heyde norming

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A natural sufficient condition is given for a Galton–Watson process in a varying environment to have a single rate of growth that obtains throughout the survival set of the process. In the homogeneous process the growth rate is provided by the usual Seneta–Heyde norming.

branching processes * varying environments

1. Introduction

We consider the Galton–Watson process in varying environments $\{Z_n\}$ defined by

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, \quad n \geq 0,$$

where $\{X_{n,i}; i\}$ are independent identically distributed copies of a random variable X_n . We let the offspring mean in the n th generation be μ_n ; thus $\mu_n = EX_n$. Then

$$E(Z_n | Z_k = 1) = \prod_{i=k}^{n-1} \mu_i \quad \text{for } n > k \geq 0,$$

and, in particular, letting $m_n = EZ_n$, we see that $m_n = \prod_{i=0}^{n-1} \mu_i$. The aim of this paper is to give a sufficient condition for $\{Z_n\}$ to have exactly one rate of growth, in the sense that

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there is a sequence of constants $\{C_n\}$ such that Z_n/C_n converges to a finite limit that is strictly positive on the survival set, $\{Z_n \rightarrow \infty\}$. It is worth noting that this is ‘one rate of growth’ in the strongest sense. A weaker sense would be for the limit of Z_n/C_n to be strictly positive on only part of the survival set, but with no other normalization giving a finite positive limit on any part of the remainder. The example in MacPhee and Schuh (1983) shows that it is, in general, possible for $\{Z_n\}$ to exhibit more than one rate of growth.

We will need a tail condition on the offspring distributions and a growth condition on the product of the offspring means. These are supplied through the following definitions. We shall say that a random variable X *dominates* $\{X_n/\mu_n\}$ if

$$P(X > x) \geq P(X_n/\mu_n > x) \quad \text{for all } x,$$

and that the process is *uniformly supercritical* if

$$\prod_{j=k}^{n+k-1} \mu_j \geq Bc^n \quad \text{for some } B > 0, c > 1, \text{ and all } n, k \geq 0.$$

Without loss of generality we take $B < 1$; note that $\inf_n \mu_n \geq B > 0$.

Let W be the limit of the nonnegative martingale $\{Z_n/m_n\}$. The main result in D’Souza and Biggins (1992) was the following theorem.

Theorem 1. *Suppose the process is uniformly supercritical and $\{X_n/\mu_n\}$ is dominated by X with $EX \log^+ X < \infty$, then $EW = 1$ and $\{W > 0\} = \{Z_n \rightarrow \infty\}$ almost surely. \square*

Thus, under these conditions, the process has a single rate of growth, given by the product of the offspring means, $\{m_n\}$. Though it is tangential to our main development, it is worth drawing attention here to the recent work of Cohn and Jagers (1992) dealing with the analogous problem for the general branching process.

Theorem 1 yields one half of the Kesten–Stigum theorem, see Asmussen and Hering (1981, II.2), when specialized to the homogeneous case. A natural conjecture is that if the dominating random variable X has finite mean then there is only one rate of growth, which obtains throughout the survival set, and this growth rate agrees with $\{m_n\}$ when $EX \log^+ X < \infty$. Here we establish that this is in fact so. The only candidates for suitable normalizing sequences can be obtained from the generating functions of $\{Z_n\}$, following the original construction of the Seneta–Heyde constants, as can be seen from Goettge (1976, Theorems 16, 17) or Schuh and Barbour (1977, Lemma 1.1.7). Under the (rather unnatural?) condition that certain generating functions commute Goettge (1976, Theorem 26) shows that these normalizing sequences do indeed provide the growth rate throughout the survival set. The following, rather weak, partial result, which was also given in D’Souza and Biggins (1992), will play a critical role in the proof here.

Theorem 2. *Suppose the process is uniformly supercritical and $\{X_n/\mu_n\}$ is dominated by X with $E(X) < \infty$, then $(Z_n/m_n)^{1/n} \rightarrow 1$ almost surely on $\{Z_n \rightarrow \infty\}$.*

Therefore the possible rates of growth of Z_n cannot differ too much from $\{m_n\}$. This

theorem was proved using Theorem 1 above. This route leaves something to be desired from the aesthetic point of view on two counts. Firstly Theorem 2 above is less deep than the result used to derive it. Secondly Theorem 1 should really be a corollary of the main result here (through Lemma 4(iii) below) but, on the present arrangement, it would play a part in the latter's derivation. We remedy this defect in the next section, giving a short proof of Theorem 2 that is independent of Theorem 1.

The main result here is the following refinement (i.e. same hypotheses) of Theorem 2. (Theorem 1 of D'Souza and Biggins, 1992, ensures that under the conditions of this theorem Z_n goes either to zero or infinity.)

Theorem 3. *Suppose that the process is uniformly supercritical and that $\{X_n/\mu_n\}$ is dominated by X with $EX < \infty$, then there exists a sequence of constants $\{C_n\}$ such that Z_n/C_n converges to a finite random variable W_{SH} with $\{W_{SH} = 0\} = \{Z_n \rightarrow 0\}$. Furthermore $\{C_n\}$ can be obtained recursively by*

$$C_{n+1} = C_n EX_n I(X_n \leq \mu_n C_n) \quad (n \geq 0)$$

provided only that C_0 is taken sufficiently large.

Two norming sequences are called *equivalent* if the ratio of corresponding terms tends to a finite positive limit. Notice that $\{C_n\}$ as defined above depends on the choice of C_0 . Therefore it is implicit in the statement (and explicit in Lemma 4(ii)) that the sequences resulting from different (sufficiently large) C_0 are equivalent. Sometimes it will be useful to be explicit about the dependence on C_0 , so, if $C_0 = a$, we may write the sequence as $\{C_n(a)\}$.

We shall assume the hypotheses of Theorem 3 hold throughout. The main proof follows a rather different course from that in D'Souza and Biggins (1992) being motivated by the truncation methods used in Asmussen and Hering (1981, II.5). It has two main stages. First we show that, for a sufficiently large, $\{C_n(a)\}$ is the maximal rate of growth for Z_n . That is to say that $\{Z_n/C_n(a)\}$ has a finite limit that is strictly positive with a positive probability. This is done by considering a suitable truncation of the original process, thereby making possible estimates based on variances. The severity of the truncation will depend on a ($= C_0$), and it turns out that its effect can be made very small by taking a sufficiently large. This part is a straightforward extension of the arguments in Asmussen and Hering (1981). The second stage of the proof is to show that the growth rate identified is in fact appropriate throughout the survival set, by showing that the sets $\{Z_n/C_n \rightarrow 0\}$ and $\{Z_n \rightarrow 0\}$ agree, almost surely. The idea for this part of the proof is similar to that used in Asmussen and Hering (1981, Proposition II.1.4) and in D'Souza and Biggins (1992). Let

$$w_k = P(Z_n/C_n \rightarrow 0 | Z_k = 1) ;$$

then $w_n^{Z_n}$ is a bounded martingale so that

$$w_0 = P(Z_n \rightarrow 0) + E \lim_{n \rightarrow \infty} w_n^{Z_n} I(Z_n \rightarrow \infty) .$$

Consequently, the desired result will follow if we show that $w_n^{Z_n} \rightarrow 0$ on $\{Z_n \rightarrow \infty\}$. In the homogeneous case, considered by Asmussen and Hering (1981), this follows once w_k is seen to be independent of k . In proving Theorem 1 the desired conclusion was obtained by showing that $\{w_n\}$ was bounded above by some $\eta < 1$. Here matters seem more delicate, for the bound on $\{w_n\}$ we have to work with is not uniform in n , but suffices when combined with the information on the growth of $\{Z_n\}$ in Theorem 2.

The next section gives the new proof of Theorem 2. The third section concentrates on the properties of the normalizing sequences. Armed with these results, the main proof is undertaken in the fourth section. A final short section draws attention to some known results about the distribution of the limit W_{SH} .

2. Proof of Theorem 2

Recall that $B < 1$ and $c > 1$ are constants appearing in the definition of uniform supercriticality. For the rest of the paper take $\varepsilon > 0$ but small enough that

$$d := c(1 - \varepsilon) > 1,$$

and now take a sufficiently large that

$$EXI(X > aB) \leq \varepsilon.$$

To prove Theorem 2 we introduce a truncated version of the original process where from generation k onwards the original offspring distribution sequence, $\{X_r: r \geq k\}$, is replaced by $\{X_r I(X_r \leq \mu_r aB): r \geq k\}$. Denote the process so obtained by $\{Z_{n,k}\}$. (Note that the process starts from a single 0th generation ancestor and agrees with $\{Z_n\}$ up to $n = k$.) It is then easy to show that $EZ_{n,k} \geq m_n(1 - \varepsilon)^n$ (cf. (2.3) of D'Souza and Biggins, 1992). Thus

$$\Delta := \liminf_{n \rightarrow \infty} \frac{Z_n}{m_n(1 - \varepsilon)^n} \geq \liminf_{n \rightarrow \infty} \frac{Z_{n,k}}{EZ_{n,k}} = \frac{1}{m_k} \sum_{i=1}^{Z_k} W_{i,k},$$

where $\{W_{i,k}: i\}$ are the (independent) limits of the normalized truncated processes initiated by the members of the k th generation.

Let $v_n = \text{Var}\{X_n/\mu_n\}$, which may of course be infinite. A simple calculation (see Fearn, 1971, or D'Souza and Biggins, 1992) reveals that

$$\text{Var}\{W\} = \sum_{n=0}^{\infty} \frac{\text{Var}\{X_n\}}{EZ_n \mu_n^2} = \sum_{n=0}^{\infty} \frac{v_n}{m_n}.$$

Suppose that the sequence $\{v_n\}$ is bounded by V then, as $m_n \geq Bc^n$,

$$\text{Var}\{W\} \leq \frac{cV}{B(c-1)}$$

and so, as W is a martingale limit, $EW = 1$. Therefore, using the simple inequality (2.1) in D'Souza and Biggins (1992),

$$P(W=0) \leq \frac{\text{Var}\{W\}}{\text{Var}\{W\}+1} \leq \frac{cV}{cV+B(c-1)} < 1.$$

Consequently, for a suitable $\eta < 1$, $P(W_{i,k}=0) \leq \eta$ for all k .

Now we see that

$$\begin{aligned} E(I(\Delta > 0) | Z_k) &\geq P\left(\frac{1}{m_k} \sum_{i=1}^{Z_k} W_{i,k} > 0 \middle| Z_k\right) \\ &= 1 - \prod_{i=1}^{Z_k} P(W_{i,k}=0) \\ &\geq 1 - \eta^{Z_k}. \end{aligned}$$

Letting $k \rightarrow \infty$, shows that, almost surely, $I(\Delta > 0) \geq I(Z_k \rightarrow \infty)$, so

$$\liminf_{n \rightarrow \infty} \left(\frac{Z_n}{m_n}\right)^{1/n} \geq 1 - \varepsilon \quad \text{on } \{Z_k \rightarrow \infty\}.$$

Finally, as $\{Z_n/m_n\}$ is a positive martingale, $\limsup(Z_n/m_n)^{1/n} \leq 1$, completing the proof. \square

3. Properties of the norming sequence

We start with a simple lemma showing that the norming constants grow at least geometrically. This is followed by two lemmas estimating certain sums arising in subsequent proofs.

We will denote the distribution function of the dominating variable X by F . The definitions of a and d are at the start of the previous section.

Lemma 1.

- (i) $\inf\{C_n(a)\} \geq aB$,
- (ii) $C_n(a) \geq am_n(1-\varepsilon)^n$,
- (iii) $C_{n+k}(a)/C_n(a) \geq Bd^k$,
- (iv) $C_{n+k} \geq C_n$ for $k \geq -\log B/\log d$.

Proof. As $\{X_n/\mu_n\}$ is dominated by X ,

$$\begin{aligned} C_{n+1} &= C_n \mu_n E(X_n/\mu_n) I((X_n/\mu_n) \leq C_n) \\ &= C_n \mu_n [1 - E(X_n/\mu_n) I((X_n/\mu_n) > C_n)] \\ &\geq C_n \mu_n [1 - EX I(X > C_n)]. \end{aligned}$$

Assume now that $C_k \geq aB$ for $k \geq n$, then

$$\begin{aligned}
C_{n+1} &\geq C_0 \left(\prod_{k=0}^n \mu_k \right) (1-\varepsilon)^{n+1} \\
&= am_{n+1} (1-\varepsilon)^{n+1} \\
&\geq aBc^{n+1} (1-\varepsilon)^{n+1} \geq aBd^{n+1} \geq aB.
\end{aligned}$$

proving (i) and then (ii). Similarly

$$C_{n+k} \geq C_n \left(\prod_{j=n}^{n+k-1} \mu_j \right) (1-\varepsilon)^k \geq C_n Bc^k (1-\varepsilon)^k = C_n Bd^k,$$

proving (iii) and (iv). \square

Lemma 2.

$$\begin{aligned}
\text{(i)} \quad & \sum_{n=0}^{\infty} C_n P(X > C_n) \leq \frac{d\varepsilon}{B(d-1)}, \\
\text{(ii)} \quad & \sum_{n=0}^{\infty} \frac{1}{C_n} EX^2 I(X \leq C_n) \leq \frac{dE(X)}{B(d-1)}.
\end{aligned}$$

Proof. Let $M = \sup\{n: C_n < x\}$ (obviously M depends on x). Then, using Lemma 1,

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n P(X > C_n) &= \sum_{n=0}^{\infty} C_n \int_{C_n}^{\infty} dF(x) \\
&= \int_{\inf\{C_n\}}^{\infty} \sum_{C_n < x} C_n dF(x) \\
&\leq \int_{aB}^{\infty} C_M \sum_{n=0}^M \frac{C_n}{C_M} dF(x) \\
&\leq \int_{aB}^{\infty} x \sum_{n=0}^M \frac{1}{Bd^{M-n}} dF(x) \\
&\leq \frac{d\varepsilon}{B(d-1)}.
\end{aligned}$$

The proof of (ii) is similar. Let $N = \inf\{n: C_n \geq x\}$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{C_n} EX^2 I(X \leq C_n) &= \int_0^{\infty} x^2 \sum_{C_n \geq x} \frac{1}{C_n} dF(x) \\
&\leq \int_0^{\infty} x^2 \frac{1}{C_N} \sum_{n=N}^{\infty} \frac{C_N}{C_n} dF(x) \\
&\leq \int_0^{\infty} x \sum_{n=0}^{\infty} \frac{1}{Bd^n} dF(x) \\
&\leq \frac{dE(X)}{B(d-1)}. \quad \square
\end{aligned}$$

Lemma 3.

$$(i) \quad \sum_{n=0}^{\infty} EXI(C_n < X \leq C_{n+k}) \leq \varepsilon \left(k+1 - \frac{\log B}{\log d} \right).$$

(ii) For $l > 1$,

$$\sum_{n=0}^{\infty} EXI(C_n < X \leq lC_n) \leq \varepsilon \left(\frac{\log l - \log B}{\log d} \right).$$

Proof. Let M and N be as defined in Lemma 2. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} EXI(C_n < X \leq C_{n+k}) &= \int_{\inf\{C_n\}}^{\infty} \left\{ \sum_{C_n < x \leq C_{n+k}} 1 \right\} x dF(x) \\
&\leq \int_{aB}^{\infty} \{M - (N-k) + 1\} x dF(x).
\end{aligned}$$

Now, by Lemma 1(iv),

$$C_{N+j} \geq C_N \quad \text{for } j \geq -\log B / \log d,$$

so

$$M \leq N - \log B / \log d$$

and therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} EXI(C_n < X \leq C_{n+k}) &\leq \int_{aB}^{\infty} \left(k+1 - \frac{\log B}{\log d} \right) x dF(x) \\
&\leq \varepsilon \left(k+1 - \frac{\log B}{\log d} \right).
\end{aligned}$$

The proof of (ii) is similar, but relies on the estimate

$$\sum_{n=0}^{\infty} I(C_n < x \leq lC_n) \leq \frac{\log l - \log B}{\log d}. \quad \square$$

In the second part of the main proof we will need to consider other norming sequences, defined like $\{C_n\}$ but with a slightly different truncation of the offspring distributions. Let

$$D_0^{(k)} = a, \quad D_{n+1}^{(k)} = D_n^{(k)} EX_{n+k} I(X_{n+k} \leq \mu_{n+k} C_n), \quad n \geq 0.$$

Obviously, as with $\{C_n\}$, $\{D_n^{(k)}\}$ depends on the starting value a . The final lemma of this section establishes that these, and other, sequences are equivalent. In particular the second part establishes that the norming sequences for different values of $\{C_0\}$ are equivalent.

Lemma 4. (i) For $k \geq -\log B / \log d$ and a sufficiently large, $\{D_n^{(k)} / C_{n+k}\}$ decreases and is bounded below by

$$Ka e^{-4\epsilon k} / C_k,$$

where K does not depend on k and is strictly positive.

(ii) For $b > a$, $\{C_n(a) / C_n(b)\}$ decreases to a strictly positive limit.

(iii) If $EX \log^+ X < \infty$ then $\{C_n(a) / m_n\}$ decreases to a strictly positive limit.

Proof. For $n \geq 0$ let $L_n = D_n^{(k)} / C_{n+k}$. Then

$$\frac{L_{n+1}}{L_n} = \frac{D_{n+1}^{(k)} / D_n^{(k)}}{C_{n+k+1} / C_{n+k}} = \frac{EX_{n+k} I(X_{n+k} \leq \mu_{n+k} C_n)}{EX_{n+k} I(X_{n+k} \leq \mu_{n+k} C_{n+k})} \leq 1,$$

by Lemma 1(iv), so L_n decreases to a limit. Furthermore,

$$\begin{aligned} L_{n+1} &= L_0 \prod_{j=0}^n \left\{ \frac{EX_{j+k} I(X_{j+k} \leq \mu_{j+k} C_j)}{EX_{j+k} I(X_{j+k} \leq \mu_{j+k} C_{j+k})} \right\} \\ &= L_0 \prod_{j=0}^n \left\{ 1 - \frac{EX_{j+k} I(C_j < X_{j+k} / \mu_{j+k} \leq C_{j+k})}{EX_{j+k} (1 - I(X_{j+k} > \mu_{j+k} C_{j+k}))} \right\} \\ &\geq L_0 \prod_{j=0}^n \left\{ 1 - \frac{EXI(C_j < X \leq C_{j+k}) + C_{j+k} P(X > C_{j+k})}{(1 - \varepsilon)} \right\}. \end{aligned}$$

As

$$\begin{aligned} \frac{EXI(C_j < X \leq C_{j+k}) + C_{j+k} P(X > C_{j+k})}{(1 - \varepsilon)} &\leq \frac{EXI(X > aB)}{(1 - \varepsilon)} \\ &\leq \frac{\varepsilon}{(1 - \varepsilon)}, \end{aligned}$$

we can choose ε small enough that $\log(1 - s) > -2s$ for $0 < s < \varepsilon / (1 - \varepsilon)$. Therefore taking logarithms gives

$$\begin{aligned}\log L_{n+1} &\geq \log L_0 - \frac{2}{1-\varepsilon} \sum_{j=0}^n \{EXI(C_j < X \leq C_{j+k}) + C_{j+k}P(X > C_{j+k})\} \\ &\geq \log L_0 - \frac{2\varepsilon}{1-\varepsilon} \left(k+1 - \frac{\log B}{\log d} + \frac{d}{B(d-1)} \right),\end{aligned}$$

where we have used Lemmas 2(i) and 3(i). Thus, as $L_0 = a/C_k$, we see that, for small ε ,

$$\lim_{n \rightarrow \infty} \frac{D_n^{(k)}}{C_{n+k}} \geq \frac{a}{C_k} K e^{-4\varepsilon k},$$

completing the proof of (i).

The proof of (ii) follows an identical pattern, with L_n now equal to $C_n(a)/C_n(b)$, to show

$$\begin{aligned}\log L_{n+1} &\geq \log L_0 \\ &\quad - \frac{2}{1-\varepsilon} \sum_{j=0}^n \{EXI(C_j(a) < X \leq C_j(b)) + C_j(b)P(X > C_j(b))\}.\end{aligned}$$

Now, using the monotonicity of $\{L_n\}$ and Lemma 3(ii),

$$\begin{aligned}\sum_{j=0}^n EXI(C_j(a) < X \leq C_j(b)) &= \sum_{j=0}^n EXI(C_j(a) < X \leq C_j(a)/L_j) \\ &\leq \sum_{j=0}^n EXI(C_j(a) < X \leq C_j(a)/L_{n+1}) \\ &\leq \left(\frac{-\log L_{n+1} - \log B}{\log d} \right) \varepsilon.\end{aligned}$$

Consequently,

$$\log L_{n+1} \geq \log L_0 - \frac{2\varepsilon}{1-\varepsilon} \left\{ \frac{-\log L_{n+1} - \log B}{\log d} + \frac{d}{B(d-1)} \right\},$$

so, rearranging this,

$$\left\{ 1 - \left(\frac{2\varepsilon}{1-\varepsilon} \right) \frac{1}{\log d} \right\} \log L_{n+1} \geq \log L_0 - \frac{2\varepsilon}{1-\varepsilon} \left\{ \frac{-\log B}{\log d} + \frac{d}{B(d-1)} \right\}$$

and letting $n \rightarrow \infty$ completes the proof of (ii).

The proof of (iii) also follows a similar pattern, except that now

$$L_{n+1} \geq L_0 \prod_{j=0}^n \{1 - EXI(X > C_j)\}$$

and a calculation like those in Lemmas 2 and 3 establishes that

$$\sum_{j=0}^{\infty} EXI(X > C_j) < \infty \quad \text{if } EX \log^+ X < \infty. \quad \square$$

4. Proof of Theorem 3

We start by introducing a branching process in varying environments starting from a single k th generation person, with the offspring distributions being given by a truncation of the original ones. Specifically the process $\{\hat{Z}_n^{(k)}\}$ is defined by

$$\hat{Z}_0^{(k)} = 1, \quad \hat{Z}_{n+1}^{(k)} = \sum_{i=1}^{\hat{Z}_n^{(k)}} X_{n+k,i} I(X_{n+k,i} \leq \mu_{n+k} C_n), \quad n \geq 0.$$

The superscript (k) will be used to indicate the starting generation and will be dropped when $k=0$. The severity of the truncation depends on a through C_n . It is easy to see that

$$E\hat{Z}_n^{(k)} = \prod_{j=0}^{n-1} \frac{D_{j+1}^{(k)}}{D_j^{(k)}} = \frac{D_n^{(k)}}{a},$$

so, by the martingale convergence theorem, $(\hat{Z}_n^{(k)})/D_n^{(k)} \rightarrow \hat{W}^{(k)}$, where $\hat{W}^{(k)}$ is a finite random variable with expectation less than or equal to one.

Recall that, by definition,

$$w_k = P(Z_{n+k}/C_{n+k} \rightarrow 0 | Z_k = 1)$$

so, in the light of the equivalence of $\{D_n^{(k)}\}$ and $\{C_{n+k}\}$ proved in Lemma 4(i),

$$w_k = P(Z_{n+k}/D_n^{(k)} \rightarrow 0 | Z_k = 1) \leq P(\hat{Z}_n^{(k)}/D_n^{(k)} \rightarrow 0) = P(\hat{W}^{(k)} = 0).$$

Consequently the following lemma provides a bound for $\{w_k\}$, as well as establishing that we have identified a growth rate for the truncated processes.

Lemma 5. For $k \geq -\log B/\log d$ there is a $\kappa > 0$ independent of k such that

$$P(\hat{W}^{(k)} > 0) \geq \kappa d^k / (C_k e^{4\epsilon k}).$$

Proof. Applying the formula for the variance of W given in Section 2, and Lemmas 4(i), 1(iii) and 2, we see that

$$\begin{aligned} \text{Var}\{\hat{W}^{(k)}\} &= \sum_{n=0}^{\infty} \frac{\text{Var}\{X_{n+k} I(X_{n+k} \leq C_n \mu_{n+k})\}}{E\hat{Z}_n^{(k)} (EX_{n+k} I(X_{n+k} \leq C_n \mu_{n+k}))^2} \\ &\leq a \sum_{n=0}^{\infty} \frac{EX_{n+k}^2 I(X_{n+k} \leq C_n \mu_{n+k})}{D_n^{(k)} (EX_{n+k} I(X_{n+k} \leq C_n \mu_{n+k}))^2} \\ &\leq a \left\{ \sum_{n=0}^{\infty} \frac{\{EX^2 I(X \leq C_n) + C_n^2 P(X > C_n)\}}{D_n^{(k)} (1-\epsilon)^2} \right\} \\ &\leq \frac{C_k e^{4\epsilon k}}{K(1-\epsilon)^2} \sum_{n=0}^{\infty} \frac{EX^2 I(X \leq C_n) + C_n^2 P(X > C_n)}{C_{n+k}} \end{aligned}$$

$$\begin{aligned} &\leq \frac{C_k e^{4\epsilon k}}{K(1-\epsilon)^2 B d^k} \sum_{n=0}^{\infty} \frac{EX^2 I(X \leq C_n) + C_n^2 P(X > C_n)}{C_n} \\ &\leq \tilde{\kappa} C_k e^{4\epsilon k} / d^k, \end{aligned}$$

With $\tilde{\kappa}$ independent of k . Standard martingale theory therefore gives $E\hat{W}^{(k)} = 1$. The result now follows as

$$P(\hat{W}^{(k)} > 0) \geq \left\{ 1 + \frac{\text{Var}\{\hat{W}^{(k)}\}}{(E\hat{W}^{(k)})^2} \right\}^{-1} \geq \left\{ 1 + \frac{\tilde{\kappa} C_k e^{4\epsilon k}}{d^k} \right\}^{-1} \geq \frac{\kappa d^k}{C_k e^{4\epsilon k}},$$

for suitable $\kappa > 0$. \square

This lemma also holds when $k=0$ as then $D_n^{(0)} = C_n$, so Lemma 4(i) is not needed. This establishes that $\{C_n\}$ is a growth rate for $\{\hat{Z}_n\}$. The next proposition shows that this truncated process can be made to agree with the original process except on an arbitrarily small part of the sample space, by taking a sufficiently large. This and the equivalence of the sequences $\{C_n(a)\}$ for different a proved in Lemma 4(ii) will complete the first part of the proof of the main theorem. The fact that $\{C_n\}$ is a rate of growth can also be established by an application of Lemma 2 of MacPhee and Schuh (1983); however, that it is in fact the maximal growth rate, in that $P(Z_n/C_n \rightarrow \infty) = 0$, must then be established separately.

Proposition 1. *For a sufficiently large, $Z_n/C_n(a)$ converges to a finite limit that is not identically zero, almost surely.*

Proof. Let

$$G_a = \{Z_n = \hat{Z}_n \text{ for all } n\},$$

where, obviously, \hat{Z}_n is the truncation based on taking $C_0 = a$. When $b > a$, the truncation based on taking $C_0 = b$ is less severe than that for $C_0 = a$ and so $G_a \subseteq G_b$. Furthermore,

$$\begin{aligned} 1 - P(G_a) &= \sum_{n=0}^{\infty} P(Z_n = \hat{Z}_n, Z_{n+1} \neq \hat{Z}_{n+1}) \\ &\leq \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} P(Z_n = \hat{Z}_n = j) j P(X_n > \mu_n C_n) \\ &\leq \sum_{n=0}^{\infty} E\hat{Z}_n P(X > C_n) \\ &= (1/a) \sum_{n=0}^{\infty} C_n P(X > C_n) \\ &\leq \frac{d\epsilon}{aB(d-1)} \rightarrow 0 \quad \text{as } a \rightarrow \infty. \end{aligned}$$

By Lemma 4(ii), $\{C_n(a)\}$ and $\{C_n(b)\}$ are equivalent, so, on G_b , (with the truncation based on $C_0=b$),

$$Z_n/C_n(a) = \hat{Z}_n/C_n(a),$$

and the latter converges to a finite limit, which is not identically zero by Lemma 5. As b is arbitrary the proof is complete. \square

The final proposition finishes the proof of Theorem 3 by showing that the growth rate $\{C_n\}$ prevails throughout the survival set.

Proposition 2. *The sets $\{Z_n/C_n \rightarrow 0\}$ and $\{Z_n \rightarrow 0\}$ agree, almost surely.*

Proof. Most of the work for this result has already been done. As noted in the introduction we must show that $w_n^{Z_n} \rightarrow 0$ on $\{Z_n \rightarrow \infty\}$. But

$$w_k \leq P(\hat{W}^{(k)} = 0) \leq 1 - \kappa d^k / (C_k e^{4\epsilon k}),$$

using Lemma 5. Therefore

$$\begin{aligned} \log w_n^{Z_n} &= Z_n \log w_n \leq Z_n \frac{-\kappa d^n}{C_n e^{4\epsilon n}} \\ &\leq \frac{-Z_n \kappa d^n}{m_n (1-\epsilon)^n e^{4\epsilon n}} \\ &= -\kappa \left(\frac{d e^{-4\epsilon}}{1-\epsilon} \right)^n \left(\frac{Z_n}{m_n} \right) \end{aligned}$$

and, by Theorem 2, this tends to minus infinity on $\{Z_n \rightarrow \infty\}$, provided only that ϵ is sufficiently small. \square

5. The distribution of W_{SH}

When the hypotheses of Theorem 3 are valid we can use the results of Klebaner and Schuh (1982) and Cohn and Hering (1983) to study properties of the random variables W_{SH} and

$$M = \sup_n \frac{Z_n}{C_n}.$$

Klebaner and Schuh (1982) show that $EM < \infty$ if and only if $EW_{SH} < \infty$. Theorem 2.4 of Cohn and Hering (1983) asserts that when $C_n/m_n \rightarrow 0$ as $n \rightarrow \infty$ we must have $EW_{SH} = \infty$ and $E \sup(Z_n/m_n) = \infty$. (This clearly implies that $EM = \infty$.) Thus, we can conclude that if the conditions of Theorem 1 are satisfied we must have $EM < \infty$ (as well as having $EW = 1$). On the other hand, in Theorem 3, if $C_n/m_n \rightarrow 0$ then $EW_{SH} = EM = \infty$.

We can also look at the distribution of W_{SH} . It is known that in the homogeneous process

W_{SH} has a strictly positive density on $(0, \infty)$, and so its distribution function is strictly increasing and continuous there. This can fail once varying environments are allowed, just take X_n identically equal to 2 for $n \geq 2$. However, the following positive result, due to Cohn (1982), covers many cases. Call a sequence $\{i_n\}$ *accessible* if $P(Z_n = i_n) > 0$ for all n .

Theorem 4. *Suppose that $\{C_n\}$ is a sequence of constants tending to infinity such that $\{Z_n/C_n\}$ converges almost surely to a (possibly infinite) random variable W_{SH} , with $P(0 < W_{SH} < \infty) > 0$. Then the distribution of W_{SH} is continuous and strictly increasing on $(0, \infty)$ if there are accessible sequences $\{i_n\}$ and $\{i'_n\}$ such that $\liminf(i_n/C_n) = 0$, $\limsup(i'_n/C_n) \rightarrow \infty$ and either extinction is possible from any starting generation or $\{i_n\}$ has an infinite subsequence of strictly positive terms. \square*

The condition that extinction is possible from any starting generation is not given in Cohn's version of the theorem, but it is needed for the middle term on the right-hand side of his (11) to be strictly positive.

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